

ALEXANDER POLYNOMIALS OF PERIODIC KNOTS†

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§0. INTRODUCTION

A knot K in S^3 is said to have period $n > 1$ if there is a transformation T of S^3 of order n such that K is invariant under T and the fixed point set of T is a circle B , disjoint from K . (The positive solution of the Smith conjecture implies that B is unknotted and the transformation is equivalent to the one-point compactification of rotation about the z -axis in $\mathbb{R}^3$.) This paper discusses a variety of issues relating to the Alexander polynomial of periodic knots.

Many techniques are available for determining the possible periods of a knot. The first significant results were those of Trotter [5, 27] on the periods of torus knots, obtained by analyzing possible actions on the fundamental group of the knot. Murasugi's study [15] of the Alexander polynomials of periodic knots proved especially powerful. Further work on the polynomial and Alexander ideals was done by Hillman [12]. More recently, results concerning the Jones polynomial [17, 18, 25, 26], hyperbolic structures on knot complements [1], and the geometry of 3-manifolds [7], have been applied to the study of periodic knots as well. Further references include [3, 4, 6, 8, 9, 10, 16, 20].

Nonetheless, conditions on the Alexander polynomial of a knot yield the most easily computable restrictions on the periods of a knot. As will be demonstrated, these methods continue to provide especially powerful tools.

Our concern here is the description of both necessary and sufficient conditions for a polynomial to be the Alexander polynomial of a periodic knot. The material on sufficiency is completely new. As a brief example of an application of these results, consider the knot polynomial $3t^4 - 9t^3 + 11t^2 - 9t + 3$. This is the polynomial of the knot 10_{162} , which does not have period 3 by [1]. We will show that in fact there is a period 3 knot having this Alexander polynomial.

Murasugi showed that the Alexander polynomial of a knot of period n satisfies certain conditions, described in detail in §1. Our focus is on two questions, one algebraic and one geometric: when does a polynomial satisfy the Murasugi conditions, and is a polynomial satisfying Murasugi's conditions the Alexander polynomial of a knot of period n ? Our formulation presents the conditions in a way that yields many new corollaries. In fact, we are able to eliminate many cases of possible periods of knots, which until now have only been addressed with much more subtle invariants. (Of course, as any knot polynomial occurs as the polynomial for an infinite collection of knots, our examples represent infinite families. Other methods are more restrictive.)

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This paper is organized as follows. In §1 we state the Murasugi conditions and show that when n is prime these conditions are equivalent to verifiable conditions (involving factoring polynomials in $\mathbb{Z}[\zeta][t]$ where ζ is a primitive n th root of unity).

Section 2 discusses a converse to the results described in Section 1. Whereas those results place necessary conditions on a polynomial to be the Alexander polynomial of a periodic knot, here we will show that under certain restrictions a polynomial satisfying those conditions is in fact the polynomial of a periodic knot. If a knot has period n , the linking number of K and B , λ , is uniquely determined by n and the Alexander polynomial of K . We will show that if a polynomial satisfies the Murasugi conditions with $\lambda = 1$ then there is a period n knot with that Alexander polynomial. This result should be compared to Seifert's characterization of Alexander polynomials of knots [21, 13], which states that a polynomial f is a knot polynomial if and only if f is symmetric and $f(1) = \pm 1$. It also relates to Levine's construction of links with specified link polynomials [14].

In the third section we will discuss splittings of Alexander modules of periodic knots, generalizing some of Murasugi's conditions to the higher Alexander polynomials. The observation that the first homology of the infinite cyclic cover of a complement of a knot of period n is a $\mathbb{Z}[G \times \mathbb{Z}]$ -module allows one to apply elementary representation theory.

Section 4 gives bounds on the possible periods of a knot based solely on its polynomial. In particular, it is shown that each nontrivial Alexander polynomial determines an upper bound on the possible periods of knots having that polynomial.

In §5 we present explicit examples of applications of the criteria developed earlier. Although known for some time, some of the criteria restricting the possible periods of knots have not been applied, even in the published enumerations of results concerning knots with crossing number less than 11. (See for instance Burde and Zieshang [4].)

Our work applies to periodic knots in homology spheres, as does much of the earlier work quoted. The only restriction is that $\Delta_{\bar{B}} = 1$, as defined below.

§1. SURVEY OF RESULTS

NOTATION. K will be a knot of period n , invariant under the transformation T with axis B . The quotient S^3/T is homeomorphic to S^3 and the quotient map is a branched cover. The images of K and B in the quotient are denoted $\bar{K}$ and $\bar{B}$. The linking number of K and B , which equals that of $\bar{K}$ and $\bar{B}$, is denoted λ . As K is connected, it follows that n and λ are relatively prime. (Conversely every periodic knot arises in the following manner. Let $\bar{B} \cup \bar{K}$ be a 2-component link in S^3 whose linking number is relatively prime to n . If $\bar{B}$ is unknotted, the inverse image of $\bar{K}$ in the n -fold cyclic cover branched over $\bar{B}$ gives a knot of period n .)

The Alexander polynomial of a knot J is denoted $\Delta_J(t)$. For a link L with two components, we denote the two-variable Alexander polynomial by $\Delta_L(s, t)$. Let ζ_n (or just ζ when n is understood) be the primitive n -th root of unity given by $\exp(2\pi i/n)$. Let G denote the cyclic group of order n with generator g and let $\mathbb{Z}_n = \mathbb{Z}/n\mathbb{Z}$. Finally, we let $\delta_\lambda(t) = (1 - t^\lambda)/(1 - t)$.

RESULTS. The key result concerning the polynomial of the periodic knot is that it is determined by the polynomial of the link consisting of $\bar{K}$ and $\bar{B}$. The formula, first proved in [15] (see also [12, 20]) states that

$$(*) \quad \delta_\lambda(t) \Delta_K(t) = \prod_{i=1}^n \Delta_{\bar{B} \cup \bar{K}}(\zeta^i, t).$$

We rephrase this result below. Let Δ be a knot polynomial and n be a positive integer.

Murasugi Conditions on (Δ, n) . There is a knot polynomial $\bar{\Delta}$, a polynomial $\Delta_G(g, t) \in \mathbb{Z}[G][t, t^{-1}]$, and a positive integer λ relatively prime to n , such that:

1. $\bar{\Delta}$ divides Δ .
2. $(\Delta/\bar{\Delta}) = \prod_{i=1}^{n-1} \Delta_G(\zeta^i, t)$.
3. a. $\Delta_G(g, 1) = \delta_\lambda(g)$. b. $\Delta_G(1, t) = \delta_\lambda(t)\bar{\Delta}(t)$.
4. $\Delta_G(g^{-1}, t^{-1}) = t^a g^b \Delta_G(g, t)$ for some a and b .

If Δ is the Alexander polynomial of a knot of period n then (Δ, n) satisfies the Murasugi conditions. They are obtained by letting Δ_G be the image of $\Delta_{\bar{B} \cup K}(s, t)$ in $\mathbb{Z}[G][t, t^{-1}]$ under the map that sends s to g . Condition 3 is a consequence of the first Torres condition on a link polynomial [24], which states that for a link polynomial, $\Delta_L(1, t) = \delta_\lambda(t)\Delta_{L_2}(t)$, where L_2 is the second component of L , along with the fact that $\bar{B}$ is unknotted. Condition 1 follows immediately from (*) and 3b, as does Condition 2. Finally, Condition 4 follows from the second Torres condition, which describes the symmetry properties of link polynomials.

We address the two natural questions that arise. First, given a knot polynomial Δ and an integer n , when are the Murasugi conditions satisfied, and second, if the Murasugi conditions are satisfied, is there a knot of period n with Alexander polynomial Δ ?

Conjecture. If (Δ, n) satisfies the Murasugi conditions, then Δ is the Alexander polynomial of a knot K of period n .

As evidence we show in §2:

THEOREM 1.1. *If (Δ, n) satisfies the Murasugi conditions with $\lambda = 1$, then Δ is the Alexander polynomial of a knot K of period n , with $\Delta_K = \Delta$.*

The case of $\lambda > 1$ is still open although we give some partial results in Theorem 2.1.

COROLLARY 1.2. *A knot polynomial Δ which is congruent to 1 modulo n is the Alexander polynomial of a knot of period n .*

Proof. Suppose $\Delta(t) = n \cdot h(t) + 1$. Let $\Sigma_G = 1 + g + \dots + g^{n-1}$ be the norm element of $\mathbb{Z}G$. Murasugi's conditions are satisfied with $\bar{\Delta} = \Delta$, $\lambda = 1$, and $\Delta_G(g, t) = \Sigma_G \cdot h(t) + 1$. $\square$

The question of when does (Δ, n) satisfy the Murasugi conditions is addressed below and in §4 and §5. Generalizations of the Murasugi conditions applying to other abelian knot invariants are given in §3.

In the case that n is a prime power p^r , the following congruence holds when (Δ, n) satisfies the Murasugi conditions.

Murasugi's Congruence [15]. $\Delta \equiv (\delta_\lambda(t))^{n-1} (\bar{\Delta}(t))^n \pmod{p}$, where $\equiv$ means congruent up to a multiple of $\pm t^l$.

Proof. There a map $\mathbb{Z}[\zeta_{p^r}]$ to $\mathbb{Z}_p$ given by sending ζ to 1. Hence conditions 2 and 3b combine to yield the above. $\square$

The Murasugi's conditions on (Δ, n) , although quite strong, and perhaps complete, are somewhat intractable, as they call for the construction of an unknown polynomial Δ_G in $\mathbb{Z}[G][t, t^{-1}]$. In the case that n is a prime p , there are simpler, and as we shall see equivalent, modified criteria.

Modified Murasugi Conditions on (Δ, p) . There is a knot polynomial $\bar{\Delta}$, a polynomial $f(t) \in \mathbb{Z}[\zeta_p][t, t^{-1}]$, and a number λ relatively prime to p , such that:

1. $\bar{\Delta}$ divides Δ .
2. $(\Delta/\bar{\Delta}) = \prod_{i=1}^{p-1} f^{\sigma_i}$, where the σ_i are the elements of the Galois group $\text{Gal}(\mathbb{Q}[\zeta]/\mathbb{Q})$.

Here $\sigma_i(\zeta) = \zeta^i$.

3. $f(1) = \delta_\lambda(\zeta)$.
4. $\bar{f}(t) = \zeta^a t^b f(t)$, where $\bar{f}(t)$ is the complex conjugate of $f(t^{-1})$.
5. $\Delta \equiv (\delta_\lambda(t))^{p-1} (\bar{\Delta}(t))^p \pmod{p}$.

As will be seen in §5, these modified criteria are easily checked. It is clear that they follow from the Murasugi conditions. Here is the converse.

THEOREM 1.3. *If (Δ, p) satisfies the Modified Murasugi conditions described above, then it satisfies the Murasugi conditions.*

Proof. The Rim square below is a pullback, for G a cyclic group of prime order p .

$$\begin{array}{ccc} \mathbb{Z}[G][t, t^{-1}] & \rightarrow & \mathbb{Z}[t, t^{-1}] \\ \downarrow & & \downarrow \\ \mathbb{Z}[\zeta][t, t^{-1}] & \rightarrow & \mathbb{Z}_p[t, t^{-1}] \end{array}$$

The top arrow is the augmentation. The right vertical arrow is reduction mod p . The left vertical arrow is obtained by replacing g with ζ and the bottom arrow sends ζ to 1.

The polynomials f and $\delta_\lambda \bar{\Delta}$ are in $\mathbb{Z}[\zeta][t, t^{-1}]$ and $\mathbb{Z}[t, t^{-1}]$ respectively. Since $\mathbb{Z}[\zeta]/(1 - \zeta) \cong \mathbb{Z}_p$, conditions 2 and 5 show that $f^{p-1} = (\delta_\lambda \bar{\Delta})^{p-1}$ in $\mathbb{Z}_p[t, t^{-1}]$. Since this is a UFD, $f = \delta_\lambda \bar{\Delta}$ in $\mathbb{Z}_p[t, t^{-1}]$ and so they pullback to $\Delta_G(g, t)$ in $\mathbb{Z}[G][t, t^{-1}]$. The symmetry conditions follow from the uniqueness of the pullback. $\square$

The Murasugi conditions as stated are somewhat intractable as stated when n is not a prime. We reformulate them. Let d be a positive divisor of n , let $f_d(t) = \Delta_G(\zeta_d, t)$ and $F_d(t) = \prod_{\sigma \in \text{Gal}(\mathbb{Q}[\zeta_d]/\mathbb{Q})} f_d^{\sigma}(t)$. By the Torres conditions $F_1(t) = \delta_\lambda(t) \Delta_K(t)$ and more generally for a subgroup H of G of index h , $\delta_\lambda(t) \Delta_{K/H}(t) = \prod_{d|h} F_d(t)$. Conversely, given a knot polynomial Δ and a possible period n , one considers all possible factorizations over $\mathbb{Z}[t, t^{-1}]$, $\Delta(t) = \prod F_d(t)$. Murasugi's congruence for prime power divisors of n restricts the possible $F_d(t)$. (Also $\prod_{d|h} F_d(t)$ must be a knot polynomial.) Next one factors $F_d(t)$ in $\mathbb{Z}[\zeta_d][t, t^{-1}]$ as a product of Galois conjugates $f_d^{\sigma}(t)$ satisfying the appropriate symmetry and normalization conditions. Finally, one sees if $(f_d(t)) \in \prod_{d|n} \mathbb{Z}[\zeta_d][t, t^{-1}]$ is in the image of $\mathbb{Z}[G][t, t^{-1}]$. This is easily done by using idempotents e_d in $\mathbb{Q}G[t, t^{-1}]$. (See also §3 for more on idempotents.)

One final remark. Given (Δ, p) for prime p which satisfies Murasugi's congruence, then λ is uniquely determined. (See [8].) Thus if Δ is a knot of period n , one sees that λ is uniquely determined by restricting to a $\mathbb{Z}_p$ -action where p divides n . Note that the λ 's determined by Murasugi's congruences for the prime divisor of n must coincide.

§2. SUFFICIENCY

We now address the question of the sufficiency of Murasugi's criteria. Two results are presented. The first is a proof of 1.1 which states that if a polynomial satisfies Murasugi's conditions with $\lambda = 1$, then there is a periodic knot with that polynomial. The second describes other families of polynomials for which Murasugi's criteria are sufficient.

It is clear that constructing periodic knots with specified polynomials is related to the problem of constructing 2-component links with specified link polynomials. However, as we face the geometric condition that one of the components be unknotted, the problems are distinct.

Two results should be mentioned however. First, Levine [14] proved that if a two variable polynomial satisfies the Torres conditions with $\lambda = 1$, then there is a link with that polynomial. Second, Hillman [11] showed that there are two variable polynomials which satisfied the Torres conditions but which are not link polynomials.

A second result of Levine corresponds to our second construction showing that if a polynomial satisfying the Torres conditions for $\lambda > 1$ can be factored in a specified way, then it too can be realized by a link. Our proof uses a simple companionship argument.

Preliminaries

In order to construct a knot with a specified polynomial satisfying Murasugi's conditions with $\lambda = 1$, we begin by examining the simplest possible example, the unknot, for which the corresponding link $\bar{B} \cup \bar{K}$ is the Hopf link. We also focus on the effect of surgery in the complement on the infinite cyclic cover of S^3 branched over $\bar{K}$.

Let L denote Hopf Link, consisting of two components, $\bar{B}$ and $\bar{K}$, of linking number 1 in S^3 as shown in Fig. 1. Let $G = \mathbb{Z}_n$, with generator g . Note that $H_1(S^3 - (\bar{B} \cup \bar{K})) = \mathbb{Z} \times \mathbb{Z}$, with generators m_b and m_k , the meridians of $\bar{B}$ and $\bar{K}$. There is a representation of $H_1(S^3 - (\bar{B} \cup \bar{K}))$ to $\mathbb{Z} \times G$ given by sending m_k to $(1, 0)$ and m_b to $(0, g)$. The corresponding $\mathbb{Z} \times G$ cover, M , (branched over $\bar{B}$) is homeomorphic to $D^2 \times \mathbb{R}$, and is acted on by $\mathbb{Z} \times G$, where $(1, 0)$ acts by translation by 1 in the $\mathbb{R}$ factor, and $(0, g)$ acts by rotation by $2\pi/n$ in the D^2 factor.

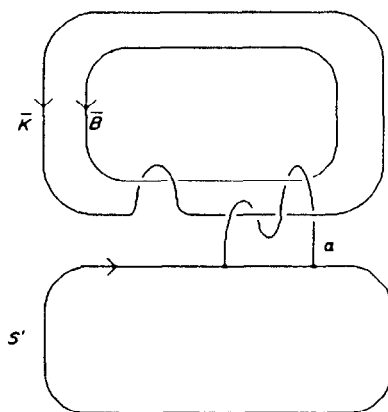


Fig. 1.

Suppose that S' is a knot in the complement of $\bar{B} \cup \bar{K}$, which is null homologous in $S^3 - (\bar{B} \cup \bar{K})$. Then the preimage of S' in M consists of a family of curves, $\mathcal{S}$, on which $\mathbb{Z} \times G$ acts freely. Hence if we denote one of those lifts by S , every element in $\mathcal{S}$ can be written uniquely as $(t^i, g^j)S$, where t is a generator of $\mathbb{Z}$, i is an integer, and $0 \leq j < n$.

The homology group $H_1(M - \mathcal{S})$ is a free $\mathbb{Z}[\mathbb{Z} \times G] = \mathbb{Z}[G][t, t^{-1}]$ module on one generator, m_S , the meridian of S .

Now suppose that surgery is performed on S' . The resulting manifold continues to have a $\mathbb{Z} \times G$ -cover, which we will denote N . (In our case the complement of $\bar{K} \cup \bar{B}$ in the

surgered manifold continues to have first homology $\mathbb{Z} \times \mathbb{Z}$, as we will perform $+1$ or -1 surgery. In general the $\mathbb{Z} \times G$ cover is obtained from the representation that sends m_k to $(1, 0)$, m_b to $(0, g)$, and the meridian of S' , $m_{S'}$, to $(0, e)$. N is obtained from M by performing $\mathbb{Z}[G][t, t^{-1}]$ equivariant surgery on $\mathcal{S}$. If integer surgery was performed on S' , then the surgery on $\mathcal{S}$ will also be integer surgery.

The homology of N is a $\mathbb{Z}[G][t, t^{-1}]$ module with one generator and one relation corresponding to the surgery. The relation is given by $f(t) = \sum a_{i,j} g^j t^i$. Here $a_{i,j} = lk(S, g^j t^i S)$. (Linking numbers in $D^2 \times \mathbb{R}$ are as usual.) The one special case is $a_{0,0}$, which is given by the surgery coefficient. It will be seen that in our situation it is determined by the other coefficients.

Altering the surgery curve S'

Let α be an embedded path in the complement of the periodic knot and its axis, meeting S' only in its end points. (See Fig. 1.) There are well defined linking numbers $\lambda_k = lk(\alpha, \bar{K})$ and $\lambda_b = lk(\alpha, \bar{B})$ obtained by computing the linking numbers of $\bar{K}$ and $\bar{B}$ with the closed path formed as the union of α and a path on S' . As S' does not link either $\bar{K}$ or $\bar{B}$ these are well defined. We will assume from now on that at least one of the two linking numbers is not 0.

Using α the curve S' can be altered as illustrated in Fig. 2. This will be called an α move of type d , where d is the number of full twists around S' . If the surgery coefficient on S' is not altered, we want to identify the effect this has on the polynomial $f(t)$.

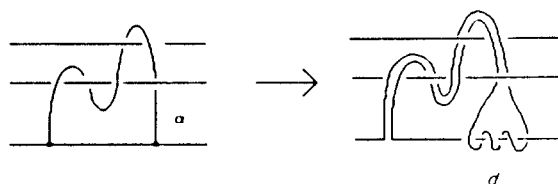


Fig. 2.

First note that α lifts to a family of arcs in M which is freely permuted by the $\mathbb{Z} \times G$ action. Let α denote the lift which is based on S . The other endpoint of α is on $(t^{\lambda_k}, g^{\lambda_b})S$. This curve is distinct from S . If the move in Fig. 2 is performed on S' , there is a corresponding alteration of the link $\mathcal{S}$. The only two linking numbers between S and another component of $\mathcal{S}$ that are affected are between S and the two components, $(t^{\lambda_k}, g^{\lambda_b})S$ and $(t^{-\lambda_k}, g^{-\lambda_b})S$. (In effect, S has reached out and grabbed $(t^{\lambda_k}, g^{\lambda_b})S$. At the same time $(t^{-\lambda_k}, g^{-\lambda_b})S$ has reached out and grabbed S .) Each of the two linking numbers is changed by d . Hence, the polynomial f is changed by adding on the term $d(g^{-\lambda_b} t^{-\lambda_k} + g^{\lambda_b} t^{\lambda_k})$. The surgery coefficient will also be changed. We will check that later. (A careful analysis at this point would show that in fact the change in the surgery coefficient is $-2d$.)

Completion of the proof of Theorem 1.1

We want to construct an example of a triple $(\bar{K}, \bar{B}, S')$ such that the resulting polynomial is some specified $\mathbb{Z}[G]$ polynomial, $\Delta_G(g, t)$. As we are in the $\lambda = 1$ case, the polynomial Δ_G is assumed to have the following properties, derived from Murasugi's conditions. First, Δ_G is symmetric; that is, that $\Delta_G(g^{-1}, t^{-1}) = \Delta_G(g, t)$. Secondly, we assume that $\Delta_G(g, 1) = 1$.

We want our example to have the property that $\bar{B} \cup S'$ is a trivial link in S^3 . Then if either $+1$ or -1 surgery is performed on S' , the resulting manifold M is again S^3 , and that $\bar{B}$ remains unknotted after the surgery is performed.

Let $k = \lfloor n/2 \rfloor$. Then Δ_G can be written as the sum $h_0(t) + gh_1(t) + \dots + g^k h_k(t) + g^{-1}h_1(t^{-1}) + \dots + g^{-k}h_k(t^{-1})$. Here the h_i are integral polynomials in $\mathbb{Z}[t, t^{-1}]$. (If n is odd this decomposition is natural. If n is even, the polynomial $h_{n/2}$ is required to be in $\mathbb{Z}[t]$.) Furthermore, h_0 is symmetric and $h_i(1) = 0$ for $i \neq 0$. The triple $(\bar{K}, \bar{B}, S')$ is illustrated in Fig. 3. There are $k+1$ boxes in that illustration, corresponding to the $k+1$ polynomials, h_i , $i = 0, \dots, k$.

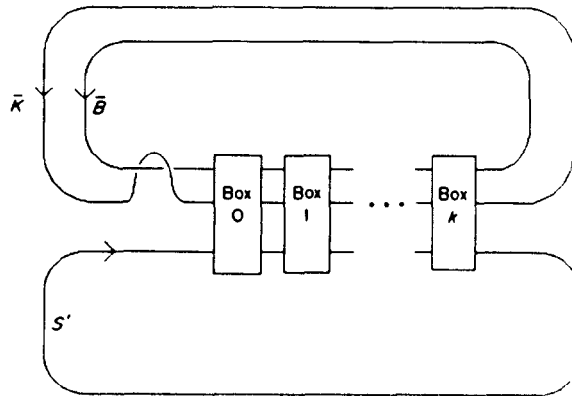


Fig. 3.

We want to describe the contents of box i . We begin with the case $i > 0$. In this case, write $h_i(t) = \sum c_j t^j$. Box i is illustrated in Fig. 4. (In the Figure $i = 3$, and $h_3 = -3t^{-3} + 2t^{-2} + 1t^2 - 2t^3 + 2t^4$.) In the box one α move has been performed for each $c_j \neq 0$. The α move is of type c_j . In addition, the j th move corresponds to an α that links $\bar{K}$ j times.

It should be clear in Fig. 4 that since $h_i(1) = 0$, if the curve $\bar{K}$ is ignored, the link $\bar{B} \cup S'$ has not been changed by the complete collection of moves in the box.

Box 0 is handled slightly differently. Write $h_0(t) = 1 + H(t) + H(t^{-1})$, where H is an integral polynomial with zero constant term. Note that $H(1) = 0$. Write $H = \sum c_j t^j$. The summation begins at $j = 1$. There is one α move for each c_j which is not 0. That α move is of type c_j and the corresponding α links $\bar{K}$ j times. Again, the condition that the sum of the c_j is 0 implies that when $\bar{K}$ is ignored the link $\bar{B} \cup S'$ is unchanged.

It is clear at this point that our construction has produced a triple $(\bar{K}, \bar{B}, S')$ for which the polynomial is the desired Δ_G , except perhaps the constant term. The constant term will turn out to be correct if one of $+1$ or -1 surgery is performed on S' . To see this, note that the $\mathbb{Z}[G][t, t^{-1}]$ polynomial that corresponds to our triple has augmentation which is the Alexander polynomial of the knot K' in the S^3 that results from the surgery (either $+1$ or -1) on S' . Hence, after augmenting, if our polynomial is evaluated at 1, the result will be either $+1$ or -1 . The polynomial Δ_G , after augmenting and evaluating at 1 yields $+1$ by assumption. This implies that our polynomial has either the correct constant term, or is off by two. Since switching from -1 to $+1$ surgery will change the constant coefficient by 2, one of the choices will yield the correct polynomial. (A careful analysis shows that the α

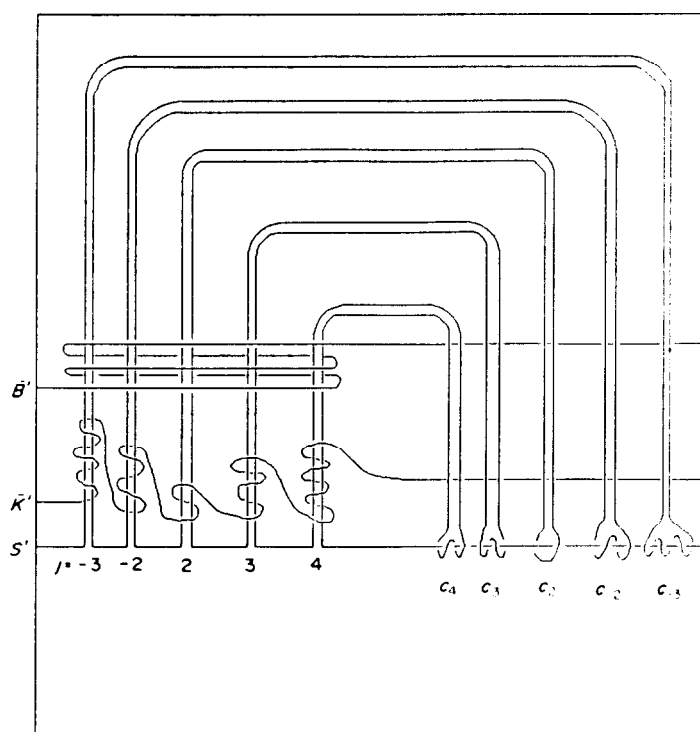


Fig. 4.

moves do not affect the value of the augmented polynomial evaluated at 1. Hence, the correct surgery coefficient is actually 1.)

Other sufficiency results

By using a companionship argument we are able to construct a variety of polynomials which are Alexander polynomials for periodic knots, with $\lambda \neq 1$.

Suppose that J a knot of period n . Removing the axis B from $S^1 \times D^2$ yields a knot J' in $S^1 \times D^2$, with winding number λ_1 . There is a natural longitude to that $S^1 \times D^2$. Now if K is a knot in S^3 , and f is a longitude preserving map of $S^1 \times B^2$ to a neighborhood of K , then $f(J')$ is a "satellite" knot with companion K . A result of Seifert [22], states that $\Delta_{f(J')}(t) = \Delta_J(t) \cdot \Delta_K(t^{\lambda_1})$.

In this situation, if K is also of period p with winding number λ_2 then $f(J')$ is also period n with winding number $\lambda_1 \cdot \lambda_2$.

Hence the following result:

THEOREM 2.1. *If Δ_1 is the polynomial of a period p knot with winding number λ_1 and Δ_2 is the polynomial of a period p knot with winding number λ_2 , then $\Delta_1(t) \cdot \Delta_2(t^{\lambda_1})$ is the polynomial of a period p knot with winding number $\lambda_1 \cdot \lambda_2$.*

Example. $\Delta(t) = (t^2 - t + 1) \cdot (3t^4 - 9t^3 + 11t^2 - 9t + 3)$ is the Alexander polynomial of a knot of period 3 with $\lambda = 2$. Indeed the first factor is the Alexander polynomial of the trefoil which has period 3 with $\lambda = 2$, and the second factor is the Alexander polynomial of a period 3 knot with $\lambda = 1$ by Corollary 1.2.

§3. ALEXANDER MODULES OF PERIODIC KNOTS

We give some further necessary conditions on Alexander modules of periodic knots in homology 3-spheres. A knot K in a homology 3-sphere Σ has period n if it is invariant under a $G = \mathbb{Z}_n$ -action on Σ with fixed set B a circle disjoint from K . Let $\bar{K} = K/G \subset \bar{\Sigma} = \Sigma/G$. Let $\pi: \tilde{X} \rightarrow X$ and $\bar{\pi}: \tilde{Y} \rightarrow Y$ be the infinite cyclic covers of the knot complements $X = \Sigma - K$ and $Y = \bar{\Sigma} - \bar{K}$. The key observation is that the G -action on $\Sigma - K$ lifts to a G -action on $\tilde{X}$ with quotient $\tilde{Y}$ and fixed set $\tilde{B} = \pi^{-1}(B)$. Indeed, let g be a generator of G . Then $g \circ \pi: \tilde{X} \rightarrow X$ induces the trivial map on H_1 and so lifts to $\tilde{g}: \tilde{X} \rightarrow \tilde{X}$. Since g has a non-empty, connected fixed-point set there is a unique lift $\tilde{g}$ with fixed points and the fixed point set is $\tilde{B}$. Since $\tilde{g}^n$ is a lift of the identity which has fixed points, it is itself the identity and hence $\tilde{g}$ is a map of period n . This gives an action of $G \times C_\infty$ on $\tilde{X}$, where C_∞ is the infinite cyclic group. Thus $H_1(\tilde{X})$ is a $\mathbb{Z}[G \times C_\infty] = \mathbb{Z}[G][t, t^{-1}]$ -module. It further follows that $\tilde{X}/G \rightarrow Y$ is an abelian cover inducing the trivial map on H_1 , so that we can identify this cover with $\bar{\pi}$ and $\tilde{X}/G$ with $\tilde{Y}$.

A transfer argument (see [2]) identifies $H_1(\tilde{X}; \mathbb{Q})^G$ with $H_1(\tilde{Y}; \mathbb{Q})$. It thus follows that the Alexander polynomials of $\bar{K}$ divide those of K . Looking in a different direction, one sees that $H_1(\tilde{X}; \mathbb{Q}[\zeta_n])$ splits as a direct sum of the eigenspaces of the action of the generator g of G . This section consists of generalizations of these simple observations.

We first examine what happens when we extend coefficients to $\mathbb{Z}[1/n]$. Let $\Phi_d(x)$ be the d -th cyclotomic polynomial. Then $x^n - 1 = \prod_{d|n} \Phi_d(x)$ is a factorization into relatively prime irreducible polynomials where the product is over all positive divisors of n . There is a natural identification of $\mathbb{Q}[x]/\Phi_d(x)$ with $\mathbb{Q}[\zeta_d]$.

LEMMA 3.1. *Let G be a cyclic group of order n with generator g , let M be a $\mathbb{Z}[1/n][G]$ -module, and $M_d = \{m \in M \mid \Phi_d(g)(m) = 0\}$. Then for H a subgroup of G of index h , $M^H = \bigoplus_{d|h} M_d$. (This includes the case where H is the trivial subgroup).*

Proof. We first consider the case of H the trivial subgroup. By taking the derivative of both sides of the equation $x^n - 1 = \prod_{d|n} \Phi_d(x)$ and dividing by nx^{n-1} we see $1 = \sum_{d|n} a_d(x) \phi_d(x)$ where $\phi_d(x) = (x^n - 1)/\Phi_d(x)$ and $a_d(x) \in \mathbb{Z}[1/n][x, x^{-1}]$ is the derivative of $\Phi_d(x)$ divided by nx^{n-1} . Substituting g for x to get an element in the group ring and applying both sides to $m \in M$ yields $m = \sum_{d|n} a_d(g) \phi_d(g)(m)$. To see that this gives the advertised direct sum decomposition, note that the d -term is in M_d since $\Phi_d(g) \phi_d(g)(m) = 0$. Also, if m is in two M_d 's then $\phi_d(g)(m)$ is zero for all d (consider the factorization of $\phi_d(x)$) and hence m is zero.

In the case of a general subgroup H , apply the above reasoning to the quotient group G/H acting on M^H . $\square$

In particular this applies to the higher Alexander ideals of $H_1(\tilde{X}; \mathbb{Z}[1/n])$. (See [4] for definitions). Let $\Delta_{k,K}(t)$ denote the k -th Alexander polynomial of the knot K ; when $k = 1$ we get "the Alexander polynomial" $\Delta_K(t)$. Applications of this idea (even when the period is 2) are given in §5.

COROLLARY 3.2. *Let K be a $G = \mathbb{Z}/n$ -periodic knot. Then for all $k \geq 1$, $\Delta_{k,K}(t) = \prod_{d|n} F_{k,d}(t)$ where $F_{k,d}(t) \in \mathbb{Z}[t, t^{-1}]$. Furthermore for H a subgroup of G of index h , $\Delta_{k,K/H}(t) = \prod_{d|h} F_{k,d}(t)$.*

Extending coefficients to $\mathbb{Z}[\zeta_n, 1/n]$ we make further progress. The d -th cyclotomic polynomial factors $\Phi_d(x) = \prod (x - \zeta_d^i)$ where the product is over all $1 \leq i \leq d$ with $(d, i) = 1$.

LEMMA 3.3. *Let G be a cyclic group with generator g , let M be a $\mathbb{Z}[1/n, \zeta_n]$ G -module, and $M_{d,i} = \{m \in M \mid (g - \zeta_d^i)(m) = 0\}$. Then M_d (defined above) is a direct sum $\bigoplus M_{d,i}$ where the sum is over all $1 \leq i \leq d$ with $(d, i) = 1$.*

Proof. It is clear that $M_{d,i}$ is contained in M_d . Let $e_{d,i} = \frac{1}{n} \sum_{j=0}^{n-1} (\zeta_d^i)^j g^{-j}$. It is easy to see that $M_{d,i} = e_{d,i} M = e_{d,i} M_d$. From the character relations for finite groups (see [23, p. 21, 50], the $e_{d,i}$ are mutually orthogonal idempotents whose sum is e , and this completes the proof. Alternatively, one can use the proof of 3.1. $\square$

If $M = V \otimes \mathbb{Z}[\zeta_n]$ where V is a $\mathbb{Z}[1/n]$ G -module one can say much more. There is an action of $\text{Gal}(\mathbb{Q}(\zeta_n)/\mathbb{Q})$ on $M_d = V_d \otimes \mathbb{Z}[\zeta_n]$ given by $\sigma(v \otimes x) = v \otimes \sigma(x)$. The action commutes with the action of G . This shows that $\text{Gal}(\mathbb{Q}(\zeta_n)/\mathbb{Q})$ permutes the summands $M_{d,i}$ for $1 \leq i \leq d$, $(d, i) = 1$. Furthermore $M_{d,i}$ is invariant under $\text{Gal}(\mathbb{Q}(\zeta_n)/\mathbb{Q}(\zeta_d))$, so there is an action of $\text{Gal}(\mathbb{Q}(\zeta_d)/\mathbb{Q})$ on the set $\{M_{d,i}\}$. Applying this to $V = H_1(\tilde{X}; \mathbb{Z}[1/n])$ one sees:

COROLLARY 3.4. *Let K be a $G = \mathbb{Z}/n$ -periodic knot. For all $k \geq 1$, let $\Delta_{k,K}(t) = \prod_{d|n} F_{k,d}(t)$ as above. Then there are polynomials $f_{k,d}(t) \in \mathbb{Z}[1/n, \zeta_d][t, t^{-1}]$ so that $F_{k,d}(t) = \prod f_{k,d}(t)^\sigma$, where the product is over all $\sigma \in \text{Gal}(\mathbb{Q}(\zeta_d)/\mathbb{Q})$.*

Remarks on integrality. While a careful look at the above proof would show that $f_{k,d}(t) \in \mathbb{Z}[1/d, \zeta_d]$, and by the Murasugi conditions, $f_{1,d}(t) \in \mathbb{Z}[\zeta_d][t, t^{-1}]$, the failure of Gauss's lemma for $\mathbb{Z}[\zeta_d]$ makes the integrality unclear for $k > 1$. However if d is a prime power p^k , the fact that there is a single prime $(\zeta_d - 1)\mathbb{Z}[\zeta_d]$ lying over $p\mathbb{Z}$ allows one to choose $f_{k,d}(t) \in \mathbb{Z}[\zeta_d]$.

§4. BOUNDS ON PERIODS

In this section we present some general consequences of Murasugi's conditions. The main results describe bounds on the periods of a knot based only on its Alexander polynomial.

THEOREM 4.1. *If K is of period p' and the leading coefficient of Δ_K is divisible by p , then it is divisible by p' .*

COROLLARY 4.2. *Let K be a knot with non-trivial Alexander polynomial Δ .*

(a) *If K has prime power period p' , then either p' divides the leading coefficient of Δ or $1 + \deg \Delta \geq p'$.*

(b) *If K has period $2'$, then either $2'$ divides the leading coefficient of Δ or $1 + (1/2)\deg \Delta \geq 2'$.*

Proof. If p' does not divide the leading coefficient then it is relatively prime to p . Thus by Murasugi's congruence, $\deg \Delta = p' \deg \Delta_{\bar{K}} + (p' - 1)(\lambda - 1)$. If $\Delta_{\bar{K}}$ is non-trivial, then we have $\deg \Delta \geq 2p'$. If it is trivial then $\lambda \neq 1$, and hence $\deg \Delta \geq (p' - 1)(2 - 1)$. For $p = 2$, we can do a little better since λ cannot be 2, so $\deg \Delta \geq (2' - 1)(3 - 1)$. $\square$

The bounds in the corollary are sharp:

Example. For p odd, the $(2, p')$ torus knot is of period p' and its Alexander polynomial, $(t^{2p'} - 1)(t - 1)/(t^{p'} - 1)(t^2 - 1)$ is monic and of degree $p' - 1$. The $(3, 2')$ torus knot has

period 2^r and Alexander polynomial $(t^{3(2^r)} - 1)(t - 1)/(t^3 - 1)(t^{(2^r)} - 1)$, which is monic and of degree $2^{(r+1)} - 2$.

Since a knot K of period n also has period p' for all prime power factors p' dividing n , the above corollary gives *a priori* bounds on the period of a knot with non-trivial Alexander polynomial. For example, suppose that Δ_K is of degree 20 and has leading coefficient $(7^3)(23)(31^2)$. Then any period of K must be a divisor of $(2^3)(3^2)(5)(7^3)(11)(13)(17)(19)(23)(31^2)$. Of course much stricter bounds can be obtained by combining criteria. For instance, as 20 cannot be expressed as $8a + 7b$, the knot could not have period 8 by Murasugi's congruence. Similarly, periods 9, 13, 17, and 19 are ruled out.

Proof of 4.1. To simplify notation, we write Δ for Δ_K , $\bar{\Delta}$ for $\Delta_{\bar{K}}$, and ξ_k for ζ_{p^k} . Let Δ_G be the polynomial with coefficients in $\mathbb{Z}[\mathbb{Z}_{p^r}]$ given by Murasugi's conditions. We normalize Δ_G so that it is a polynomial in t with nonzero constant term. For $0 \leq k \leq r$, there is a natural map from $\mathbb{Z}[\mathbb{Z}_{p^r}]$ to $\mathbb{Z}[\xi_k]$ given by $g \rightarrow \xi_k$. Denote the image of Δ_G in $\mathbb{Z}[\xi_k][t]$ by h_k . We have that $\delta_\lambda \Delta$ is the product of the h_k and their conjugates in $\mathbb{Z}[\xi_k]$:

$$\delta_\lambda \Delta = \prod_{0 \leq k \leq r} \prod_{\sigma \in \text{Gal}(\mathbb{Q}[\xi_k]/\mathbb{Q})} h_k^\sigma.$$

Note that the Galois group of $\mathbb{Q}[\xi_k]$ has $p^k - p^{k-1}$ elements. (In the degenerate case of $k = 0$, the Galois group has one element.) Also recall that $h_0 = \delta_\lambda \bar{\Delta}$.

Let $a \in \mathbb{Z}[\mathbb{Z}_{p^r}]$ be the leading coefficient of Δ_G as a polynomial in t and a_k be its image in $\mathbb{Z}[\xi_k]$. (It is possible that a_k is zero for some k , but at least one a_k is non-zero.) The leading coefficient of Δ is the leading coefficient of $\delta_\lambda \Delta$ which is the product of the leading coefficients of the h_k^σ . In particular the leading coefficient of Δ has as a factor the product of the conjugates of the non-zero a_k . Thus the algebraic lemma below completes the proof of the theorem.

ALGEBRAIC LEMMA. *Let a be an element in $\mathbb{Z}[\mathbb{Z}_{p^r}]$ and let a_k denote its image in $\mathbb{Z}[\xi_k]$. Let $N(a_k) = \prod_{\sigma \in \text{Gal}(\mathbb{Q}[\xi_k]/\mathbb{Q})} (a_k^\sigma)$ and let $A(a) = \prod_{a_k \neq 0} N(a_k)$. If $A(a)$ is divisible by p , then $A(a)$ is divisible by p^r .*

Proof. The mod p augmentation $\varepsilon: \mathbb{Z}[\mathbb{Z}_{p^r}] \rightarrow \mathbb{Z}_p$ (this is the unique ring homomorphism with $\varepsilon(g) = 1$) factors through $\varepsilon_k: \mathbb{Z}[\xi_k] \rightarrow \mathbb{Z}_p$. Since $\varepsilon_k(a_k^\sigma) = \varepsilon(a)$, we have $\varepsilon_k(N(a_k)) = \varepsilon(a)^{p^k - p^{k-1}}$. The hypothesis guarantees that $\varepsilon(a) \neq 0$, so p divides $N(a_k)$ all $k \leq r$.

Consider the commutative diagram of rings

$$\begin{array}{ccc} \mathbb{Z}[\mathbb{Z}_{p^r}] & \xrightarrow{\beta} & \mathbb{Z}[\mathbb{Z}_{p^{r-1}}] \\ \downarrow \alpha & & \downarrow \delta \\ \mathbb{Z}[\xi_r] & \xrightarrow{\gamma} & \mathbb{Z}_p[\mathbb{Z}_{p^{r-1}}] \end{array}$$

where $\alpha(g) = \xi_r$, $\beta(g) = g$, $\gamma(\xi_r) = g$ and $\delta(g) = g$. (Here, as always, g denotes the generator of the cyclic group). There are three cases: (i) $\alpha(a) \neq 0$, $\beta(a) \neq 0$, (ii) $\alpha(a) \neq 0$, $\beta(a) = 0$, and (iii) $\alpha(a) = 0$, $\beta(a) \neq 0$. In the first case $A(\alpha) = N(a_r)A(\beta(a))$ and the above comments show p divides $N(a_r)$ and the factors of $A(\beta(a))$. By induction on r we see p^{r-1} divides $A(\beta(a))$ and hence p^r divides $A(\alpha)$.

In the second case, it is not difficult to see that any element in the kernel of β is a multiple of $1 - g^{p^{r-1}}$. Hence it suffices to consider the case where $a = 1 - g^{p^{r-1}}$ and compute $A(a) = N(1 - \xi_r^{p^{r-1}})$. When $r = 1$, we see that $N(1 - \xi_1)$ augments to zero in $\mathbb{Z}_p$, and hence is divisible by p . (Actually it is equal to p but we don't use this.) For a general r note $1 - \xi_r^{p^{r-1}} = 1 - \xi_1$, and the field extension $\mathbb{Q}[\xi_r]/\mathbb{Q}$ factors as $\mathbb{Q}[\xi_1]/\mathbb{Q}$ followed by $\mathbb{Q}[\xi_r]/\mathbb{Q}[\xi_1]$. Thus $N_{\mathbb{Q}[\xi_r]/\mathbb{Q}}(1 - \xi_1) = N(1 - \xi_1)^{p^{r-1}}$ and hence is divisible by p^r .

Finally we consider the case where $\alpha(a) = a_r = 0$. Then $\beta(a) = pa'$ and $A(a) = A(\beta(pa'))$. If $a'_k \neq 0$ for all $k \leq r-1$, then $A(\beta(pa')) = p^r A(\beta(a'))$. On the other hand if $a'_k = 0$ for some $k \leq r-1$, then $\varepsilon(a') = \varepsilon_k(a') = 0$, so p divides $N(a'_d)$ for all $d \leq r-1$. By induction p^{r-1} divides $N(\beta(a'))$, so p^r divides $A(\beta(pa'))$.

§5. EXAMPLES

In this section we present a variety of examples arising from 10 crossing knots, indicating what can be concluded about their possible periods based only on their Alexander polynomials (and ideals). Refer to the Table for a list of the Alexander polynomials.

Factoring conditions

One aspect of Murasugi's conditions which seem not to have been applied in the literature is the factoring condition. Deciding how a polynomial factors over $\mathbb{Z}[\zeta_d]$ can be a difficult algebraic problem.

One method for gaining information about such factorizations is based on the fact that $\mathbb{Z}[\zeta_d]$ maps onto the finite field $\mathbb{Z}_p$ for any prime congruent to 1 mod d . (The units of $\mathbb{Z}_p$ are a cyclic group of order $p-1$, and hence contain a d -th root of unity if $p-1$ is divisible by d .) The following examples exploit this.

Example. The knots $10_{6,1}$, $10_{6,9}$, $10_{1,22}$, and $10_{1,64}$ are listed in [4] as having possible period 3. Each has since been shown to not have period 3 [25], [1], [10]. For each of these examples, Murasugi's congruence implies that $\bar{\Delta} = 1$. Hence, if any were of period 3 its polynomial would factor as the product of two conjugate polynomials over $\mathbb{Z}[\zeta_3]$, and hence would factor in $\mathbb{Z}_p$ when 3 divides $p-1$.

The quadratic factor of the polynomial for $10_{6,1}$ is irreducible mod 7. Hence the polynomial for $10_{6,1}$ can't factor into cubics. The polynomial for $10_{6,9}$ factors as the product of three irreducible quadratics mod 19, $((t^2 + 16t + 8)(t^2 + 13t + 1)(t^2 + 2t + 12))$, so it can't factor into cubics either.

For $10_{1,22}$ the quadratic formula shows that the first two quadratic factors of the polynomial do factor in $\mathbb{Z}[\zeta_3]$, but the third is irreducible. Hence this polynomial cannot be the product of two conjugate factors either.

The polynomial for $10_{1,64}$ reduces to $3(t^4 + t^3 + t^2 + t + 1)$ mod 7, which is irreducible.

As a final example of the application of the factoring conditions, consider $10_{1,05}$, which is listed as having possible period 7 in [4]. Murasugi's congruence implies that the quotient knot would have to be trivial, and hence that the Alexander polynomial would factor into 6 conjugate factors. However, this polynomial has irreducible factorization $(t^2 + 8t + 1)(t^2 + 23t + 1)(t + 11)(t + 8)$ mod 29.

Alexander module

Recall that rationally the Alexander module of a periodic knot splits as the direct sum of the Alexander module of the quotient knot and a second factor. If $(\bar{\Delta})^2$ divides the Alexander polynomial of the knot, then clearly the second Alexander polynomial $\Delta_{2,K}$ must be nontrivial.

The knots 10_{62} , 10_{65} , 10_{77} , 10_{87} , and 10_{137} , are all listed as having period 2 in [4]. In [1] it was shown that 10_{87} and 10_{137} in fact are not of period 2. Our methods show that all five do not have period 2. In each case Δ is divisible by a square polynomial. However, in all these cases, a check of Murasugi's congruence shows that the polynomial divides $\bar{\Delta}$ but that the square does not. Finally, all these knots have trivial second Alexander polynomial.

The knot 10_{143} is listed as having period 3 in [4]. However, Murasugi's congruence implies that $\bar{\Delta} = t^2 - t + 1$, so the same argument implies that the second Alexander ideal is nontrivial, again a contradiction.

Realizability examples

Each of the knots 10_{66} , 10_{95} , and 10_{162} , are listed as having possible period 3 in [4]. In each case hyperbolic methods [1] have shown that these do not have period 3. The results of Section 2 show (see below) that in each case these knots have the Alexander polynomials of period 3 knots. Also, 10_{101} has the Alexander polynomial of a period 7 knot by (1.2) but Traczyk [26] has shown that 10_{101} does not have period 7 using the Kauffman bracket polynomial (see also [1]).

The polynomial for 10_{66} factors as a product of a period 3, $\lambda = 2$, knot polynomial, and a period 3, $\lambda = 1$, knot polynomial (by 1.2). Hence apply Theorem 2.1.

For 10_{95} note that $t^4 - 3t^3 + 5t^2 - 3t + 1$ factors as

$$(t^2 + (\zeta_3 - 1)t + 1)(t^2 + (\zeta_3^2 - 1)t + 1).$$

Theorem 1.3 and 1.1 apply in this case. For 10_{162} Corollary 1.2 applies.

Table

10_{61} :	$\Delta = (t^2 - t + 1)(-2t^4 + 3t^3 - t^2 + 3t - 2)$
10_{62} :	$\Delta = (t^2 - t + 1)^2(t^4 - t^3 + t^2 - t + 1)$
10_{65} :	$\Delta = (t^2 - t + 1)^2(-2t^2 + 3t - 2)$
10_{66} :	$\Delta = (t^2 - t + 1)(-3t^4 + 6t^3 - 7t^2 + 6t - 3)$
10_{69} :	$\Delta = t^6 - 7t^5 + 21t^4 - 29t^3 + 21t^2 - 7t + 1$
10_{77} :	$\Delta = (t^2 - t + 1)^2(-2t^2 + 3t - 2)$
10_{87} :	$\Delta = (t^2 - t + 1)^2(-2t^2 + 5t - 2)$
10_{95} :	$\Delta = (2t^2 - 3t + 2)(-t^4 + 3t^3 - 5t^2 + 3t - 1)$
10_{101} :	$\Delta = 7t^4 - 21t^3 + 29t^2 - 21t + 7$
10_{105} :	$\Delta = t^6 - 8t^5 + 22t^4 - 29t^3 + 22t^2 - 8t + 1$
10_{122} :	$\Delta = (t^2 - t + 1)(t^2 - 3t + 1)(-2t^2 + 3t - 2)$
10_{137} :	$\Delta = (t^2 - 3t + 1)^2$
10_{140} :	$\Delta = (t^2 - t + 1)^2$
10_{143} :	$\Delta = (t^2 - t + 1)^3$
10_{162} :	$\Delta = 3t^4 - 9t^3 + 11t^2 - 9t + 3$
10_{164} :	$\Delta = 3t^4 - 11t^3 + 17t^2 - 11t + 3$

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